$k = \mathbb{Z}_{(p)}$

$$\begin{tikzcd} H \arrow[r] \arrow[dr] & \mathrm{Wfil} \arrow[d] \\ & \mathbb{A}^1/\mathbb{G}_m \end{tikzcd}$$

<u>where we are</u>: 
- constructed $S^1_{fil} = BH$
- analyzed $(S^1_{fil})^u \simeq B\mathrm{Fix} \simeq \mathrm{Aff}(S^1) \simeq \mathrm{Spec}^\Delta(C^*(S^1;k))$

$$(S^1_{fil})^{gr} \simeq [B\mathrm{Ker}/\mathbb{G}_m] \simeq \mathrm{Spec}^\Delta(k[\eta])$$

<u>next</u>: want to understand filtered HH as object of $\mathrm{QCoh}(BS^1_{fil})$ ⇜ "filtered $S^1$-action"

<u>now</u>: understand this category

§4.1 $R\Gamma(BS^1_{fil}; \mathcal{O}) := p_* \mathcal{O} \in QCoh(\mathbb{A}^1/\mathbb{G}_m)$

$$\begin{tikzcd} BS^1_{fil} \arrow[d, "p"] \\ \mathbb{A}^1/\mathbb{G}_m \end{tikzcd}$$

filtered $E_\infty$-algebra

**Prop** Let $k[u]$ be free filtered $E_1$-algebra on a generator of degree 2, weight $-1$.

There's an equivalence of filtered $E_1$-algebras $k[u] \to R\Gamma(BS^1_{fil}, \mathcal{O})$

Pf To make a map, just need map of filtered module.

$$k[-2](1) \longrightarrow R\Gamma(BS^1_{fil}; G)$$

$\Leftarrow$ map of filtered stacks

$$BS^1_{fil} \longrightarrow B^2\mathbb{G}_m(1)$$

define it to be $B^2$ of:

$$H \longrightarrow W^{fil} \longrightarrow \mathbb{G}_m(1)$$

$$\mathrm{Ker} \to W \to \mathbb{G}_m$$

$$\begin{tikzcd} \mathbb{Z} \arrow[rrr] \arrow[dr] & & & \mathbb{G}_m \\ & \mathrm{Fix} \arrow[r] & W \arrow[ur] & \end{tikzcd}$$

want to prove that $k[u] \longrightarrow R\Gamma(BS^1_{fil}; G)$

<u>underlying</u>: $R\Gamma((BS^1_{fil})^u, G) \simeq C^*(BS^1; k) \simeq k[u]$

$\cong$

$R\Gamma(B(S^1_{fil})^u, G)$

$\cong \lim_{\Delta} R\Gamma((S^1_{fil})^u, G)^{\otimes n}$

$\cong \lim_{\Delta} C^*(S^1; k)$

<u>assoc. gr</u> $\quad R\Gamma(B(S^1_{fil})^{gr}; G) \simeq \lim_{\Delta} R\Gamma((S^1_{fil})^{gr}; G)^{\otimes n}$

$$\simeq \lim_{\Delta} k[\eta]^{\otimes n} \qquad (*)$$

$(*)$ is dual to $\operatorname{colim}_{\Delta} k[\varepsilon]^{\otimes n}$ $\quad$ ignore coalgebra str $\quad$ deduce that (ignoring algebra str)

$$\simeq \operatorname{colim}_{\Delta} C_*(S^1; k)^{\otimes n}$$

$$\simeq C_*(BS^1; k)$$

$(*) \simeq C^*(BS^1; k)$

## §4.2 $\mathrm{QCoh}(BS^1_{\mathrm{fil}})$

**Def** $\Lambda := k[\varepsilon]$ regarded as a graded dg Hopf algebra ("strict")

$\varepsilon^2 = 0$,
$\varepsilon$ in degree $-1$, weight $1$

$\mathrm{Mod}^{\mathrm{str}}_\Lambda$ = 1-category of $\Lambda$-modules (in graded cochain complexes)

$\mathrm{Mod}_\Lambda := \mathrm{Mod}^{\mathrm{str}}_\Lambda[\mathrm{qiso}^{-1}]$ ($\infty$-category)

Hopf algebra structure induces a symmetric monoidal structure on

**Prop** (i) pullback along $\alpha : BS^1 \to B(S^1_{fil})^u \simeq \mathrm{Aff}(S^1)$,

induces an equivalence $\alpha^* : \mathrm{QCoh}(B(S^1_{fil})^u) \xrightarrow{\simeq} \mathrm{QCoh}(BS^1)$

(ii) There is a symmetric monoidal equivalence

$$\mathrm{Mod}_\Lambda \xrightarrow{\simeq} \mathrm{QCoh}(\underbrace{B(S^1_{fil})^{gr}}_{B\mathbb{G}_m}).$$

(i)
$$\begin{tikzcd}
\mathrm{QCoh}(B(S^1_{fil})^u) \arrow[r, "\alpha^*"] \arrow[d, "\wr"] & \mathrm{QCoh}(BS^1) \arrow[d, "\wr"] \\
\lim_{\Delta} \mathrm{QCoh}(((S^1_{fil})^u)^n) \arrow[r, "\alpha_n^*"] & \lim_{\Delta} \mathrm{QCoh}((S^1)^n)
\end{tikzcd}$$

QCoh takes colimits of stacks to limits of $\infty$-categories

<u>$\alpha_n$ is fully faithful</u>

$$
\begin{tikzcd}
\mathrm{QCoh}\left(\left(\left(S^1_{fil}\right)^u\right)^n\right) \arrow[rr, "\alpha_n^*"] & & \mathrm{QCoh}\left(\left(S^1\right)^n\right) \\
\mathrm{QCoh}\left(B\mathrm{Fix}^n\right) \arrow[u, "\cong"'] & & \\
\mathrm{Mod}_{R\Gamma(B\mathrm{Fix}^n;\, \mathcal{O})} \arrow[u, "\wr"'] \arrow[rr, "\sim"] & & \mathrm{Mod}_{C^*((S^1)^n;\, k)} \arrow[uu, "\beta"']
\end{tikzcd}
$$

**$\beta$ fully faithful** $\quad T = (S^1)^n$

$$\mathrm{Mod}_{C^*(T,k)} \underset{\beta'}{\overset{\beta}{\rightleftarrows}} \mathrm{QCoh}(T) \simeq \mathrm{Fun}(T, \mathrm{Mod}_k)$$

$\beta' : \mathrm{Fun}(T, \mathrm{Mod}_k) \longrightarrow \mathrm{Mod}_{C^*(T,k)}$ is given $\lim_T (-)$

**observe**: $\beta'$ preserve colimits, because $T$ is a finite CW complex

(e.g. $T = S^1 = B\mathbb{Z}$, $\lim_T$ is given by fiber of some map)

want to prove $\beta' \cdot \beta \longrightarrow \mathrm{id}$ is an equivalence $\Longleftrightarrow$ $\beta'\beta(C^*(T,k)) \simeq C^*(T,k)$

$$\begin{array}{ccc}\alpha^*: \ QCoh(B(S^1_{fil})^u) & \longrightarrow & QCoh(BS^1) \\ \wr\| & & \wr\| \\ \lim_{\Delta} QCoh((S^1_{fil})^{u,n}) & \xrightarrow{\alpha_n^*} & \lim_{\Delta} QCoh((S^1)^n) \\ & & \Downarrow \\ & & (X_n)\end{array}$$

need to show $X_n$ in essential image of $\alpha_n$.

clear for $n=0$, and all $X_n$ are base-changed from $X_0$ ✓